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ON SPERNER FAMILIES IN WHICH NO k SETS HAVE AN EMPTY INTERSECTION III

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Let R be an r-element set and $\mathcal{F}$ be a Sperner family of its subsets, that is, $X \not\subseteq Y$ for all different $X, Y \in \mathcal{F}$. The maximum cardinality of $\mathcal{F}$ is determined under the conditions 1) $c \leqq |X| \leqq d$ for all $X \in \mathcal{F}$, (c and d are fixed integers) and 2) no k sets ($k \geqq 4$, fixed integer) in $\mathcal{F}$ have an empty intersection. The result is mainly based on a theorem which is proved by induction, simultaneously with a theorem of Frankl.

1. Introduction and results

A *Sperner family* $\mathcal{F} = \{X_1, X_2, \ldots, X_n\}$ is a set of subsets of $R = \{1, 2, \ldots, r\}$ ($r \geqq 2$) such that none of the subsets contains another one. Let $k \geqq 3$ be an integer. $|\mathcal{F}|$ denotes the cardinality of $\mathcal{F}$ while $|X|$ denotes the cardinality of X.

$\mathfrak{G}(r, k)$ denotes the set of all Sperner families $\mathcal{F}$ on R satisfying $\bigcup_{j=1}^{k} X_{i_j} \neq R$ for all integers $i_1, i_2, \ldots, i_k$ with $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$.

Let c and d be integers satisfying $0 \leqq c \leqq d \leqq r$.

Let $\mathfrak{G}_{c,d}(r, k) = \{\mathcal{F}\colon \mathcal{F} \in \mathfrak{G}(r, k),\ c \leqq |X| \leqq d \text{ for all } X \in \mathcal{F}\}$. Finally let $n_{c,d}(r, k) = \max\{|\mathcal{F}|\colon \mathcal{F} \in \mathfrak{G}_{c,d}(r, k)\}$ if $\mathfrak{G}_{c,d}(r, k) \neq \emptyset$.

$\mathcal{F} = \{X_1, X_2, \ldots, X_n\} \in \mathfrak{G}_{c,d}(r, k)$ holds if and only if

$$\mathcal{F}' = \{R \setminus X_1, R \setminus X_2, \ldots, R \setminus X_n\}$$

is a Sperner family on R in which no k sets have an empty intersection, where $r - d \leqq |X| \leqq r - c$ for all $X \in \mathcal{F}'$. In this way we obtain analogous results for Sperner families in which no k sets have an empty intersection.

$n_{0,r}(r, k)$ and $n_{c,d}(r, 3)$ were studied in a paper by Frankl [5] and several papers by the author [9], [10] and [11].

In the present paper we will consider $n_{c,d}(r, k)$ with $k \geqq 4$. For some values of c and d it is fairly easy to determine $n_{c,d}(r, k)$. We formulate these results in the following propositions. They are easy consequences of the results of the papers mentioned above.

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Proposition 1 ([9], Lemma 1). $\mathfrak{G}_{c,d}(r,k)\neq\emptyset$ *(i.e.* $n_{c,d}(r,k)$ *exists) if and only if* $c\leqq r-1$.

Proposition 2 ([5], Theorem 1).

$$n_{c,d}(r,k)=\begin{cases}\binom{r-1}{c} & \text{if } c>\frac{r-1}{2},\\ \binom{r-1}{[(r-1)/2]} & \text{if } d\geqq\frac{r-1}{2}\geqq c\geqq\frac{r}{k},\\ \binom{r-1}{d} & \text{if } \frac{r-1}{2}>d\geqq c\geqq\frac{r}{k}.\end{cases}$$

Proposition 3

$$n_{c,d}(r,k)=\binom{r}{d}\quad \text{if } \frac{r-1}{k}\geqq d.$$

$n_{0,r}(r,k)=\binom{r-1}{[(r-1)/2]}$, $k\geqq 4$, was proved in Gronau [9]. Since $n_{c,d}(r,k)\leqq n_{0,r}(r,k)$ we have immediately

Proposition 4

$$n_{c,d}(r,k)=\binom{r-1}{\left[\frac{r-1}{2}\right]}\quad \text{if } d\geqq\left[\frac{r-1}{2}\right],\ \frac{r}{k}\geqq c.$$

The real problem, which we will solve in the present paper, is $\frac{r-1}{2}>d\geqq\frac{r}{k}>c$.

Theorem 1. *Let* $\frac{r-1}{2}>d\geqq\frac{r-1}{k-1}$ *and* $\frac{r}{k}>c$. *Then*

$$n_{c,d}(r,k)=\binom{r-1}{d}$$

for $r\geqq r_0(k)$ *(e.g.* $r_0(k)=7k(k-1)+1$*).*

Theorem 2. *Let* $\frac{r-1}{k-1}>d\geqq\frac{r}{k}>c$. *Then*

$$n_{c,d}(r,k)=\begin{cases}\binom{r-1}{d}+\binom{r-1}{r-(k-1)d-2} & \text{if } c\leqq r-(k-1)d-1,\\ \binom{r-1}{d} & \text{if } c\geqq r-(k-1)d,\end{cases}$$

where $d=\frac{r-1}{k-1+\varepsilon}$ *with fixed* ε, $0<\varepsilon<1$ *and* $r\geqq r_0(k,\varepsilon)$ $\left(\textit{e.g. } r_0(k,\varepsilon)=\left(\frac{5k^2}{1-\varepsilon}\right)^2+1\right)$.

One of the basic results on Sperner families in which no k sets have the union R (or an empty intersection) is the following generalization of the Erdős—Ko—Rado-Theorem [4] which is due to Frankl [5].

Theorem 3. *Let* $k \geqq 2$ *and let* $d \geqq \frac{r}{k}$*: Then*

$$n_{d,d}(r, k) = \binom{r-1}{d}.$$

We will give a very simple proof of this result using the following theorem which is a generalization of a theorem of the author, presented in [10].

Theorem 4. *Let* r, d, s, k *be given integers, satisfying* $k \geqq 2$, $r \geqq s \geqq d$, $k \cdot d \geqq s$. *If* $\mathscr{F}$ *is a family of* d*-element subsets of* R, *let* $\mathscr{G}^*_{s,d}(\mathscr{F})$ *be the family of those* s*-element subsets of* R *which are unions of at most* k *sets of* $\mathscr{F}$. *Then*

$$|\mathscr{G}^*_{s,d}(\mathscr{F})| \geqq \binom{r-1}{s-1}\left\{\frac{r-d}{d}\frac{|\mathscr{F}|}{\binom{r-1}{d}}+\frac{r}{s}-\frac{r}{d}\right\}. \tag{1}$$

Remark. $\mathscr{G}^*_{s,d}(\mathscr{F})$, in fact depends on k too, but since $|\mathscr{G}^*_{s,d,k+1})\mathscr{F})| \geqq |\mathscr{G}^*_{s,d,k}(\mathscr{F})|$ we may take $\mathscr{G}^*_{s,d}(\mathscr{F})$ for the smallest k with $k \cdot d \geqq s$ and have in this way a lower bound for all $\mathscr{G}^*$'s.

Corollary 1. *If* $|\mathscr{F}| = \binom{r}{d}$, *then* $|\mathscr{G}^*_{s,d}(\mathscr{F})| = \binom{r}{s}$.

Corollary 2. *If* $|\mathscr{F}| \geqq \binom{r-1}{d}$, *then* $|\mathscr{G}^*_{s,d}(\mathscr{F})| \geqq \binom{r-1}{s}$.

We notice that there is a family $\mathscr{F}$ with $|\mathscr{F}| = \binom{r-1}{d}$ and $|\mathscr{G}^*_{s,d}(\mathscr{F})| = \binom{r-1}{s}$, e.g. $\mathscr{F} = \{X\colon X \subseteq R,\ |X| = d,\ r \notin X\}$.

Corollary 3. *If* $|\mathscr{F}| > \frac{s-d}{s}\binom{r}{d}$, *then* $|\mathscr{G}^*_{s,d}(\mathscr{F})| \geqq 1$.

Theorem 4 was successfully used in solving a similar problem by Gronau [12].

In section 2 we will present some necessary background results. In section 3 we will prove theorems 3 and 4, while in section 4 and 5 theorems 1 and 2, respectively, will be proved.

2. Background results

If $\mathscr{F}$ is a family of subsets of R, then $\mathscr{F}_i$ $(i=0, 1, \ldots, r)$ denotes the family of sets belonging to $\mathscr{F}$ and having cardinality i, while the parameters $p_0, p_1, \ldots, p_r$ of $\mathscr{F}$ are the numbers of sets in $\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_r$, respectively.

Let $p_0, p_1, \ldots, p_r$ be the parameters of a Sperner family on R. Lubell [16], Meshalkin [17] and Yamamoto [19] proved the following inequality

$$\sum_{v=0}^{r} \frac{p_v}{\binom{r}{v}} \leqq 1,$$

which is called the *LYM inequality*.

Let $\Delta\mathscr{F}=\{X\colon X\subseteq R,\ X=Y\setminus\{v\}$ for some $Y\in\mathscr{F}$ and some $v\in Y\}$. If $|\mathscr{F}_i|\leqq \binom{s}{i}$, then from the Kruskal—Katona-Theorem [14], [15] we have

$$|\Delta\mathscr{F}_i|\geqq \frac{\binom{s}{i-1}}{\binom{s}{i}}|\mathscr{F}_i|.$$

If X and Y are two subsets of R, neither of which contains the other, write $X\prec Y$ if $\max\{v\colon v\in X\setminus Y\}<\max\{v\colon v\in Y\setminus X\}$. This defines a total order of the sets of the same cardinality. Clements [1] and Daykin, Godfrey and Hilton [2] have shown that if $\mathscr{F}_i$ consists of the first p_i sets of cardinality i, $\Delta\mathscr{F}_i$ consists of the first sets of cardinality $(i-1)$, and if there is a Sperner family $\mathscr{F}$ with parameters $p_0, p_1, p_2, \ldots, p_r$ then there is a so called canonical Sperner family $\mathscr{S}\mathscr{F}$ with the same parameters. The family $\mathscr{S}\mathscr{F}$ is as follows. If m is the largest index such that $p_m\neq 0$ we take the first p_m sets of cardinality m; then we take the first p_{m-1} sets of cardinality $m-1$ which are not contained by any sets of $(\mathscr{S}\mathscr{F})_m$; then we take the first p_{m-2} sets of cardinality $m-2$ which are not contained by any sets of $(\mathscr{S}\mathscr{F})_m\cup(\mathscr{S}\mathscr{F})_{m-1}$; and so on. Greene and Hilton [8] proved that if $\mathscr{F}$ belongs to $\mathfrak{G}(r,k)$ with parameters $p_0,p_1,\ldots,p_r$, where $p_i=0$ for all $i\leqq\frac{r-1}{k}$, then $\mathscr{S}\mathscr{F}$ belongs to $\mathfrak{G}(r,k)$, too; in fact no set in $\mathscr{S}\mathscr{F}$ contains the element r.

3. Proofs of theorems 3 and 4

3.1. If $k=2$, then Theorem 3 is the original Erdős—Ko—Rado-Theorem [4] (in the dual form).

We will prove both theorems jointly by induction on k. If Theorem 3 is proved for certain k, we imply the statement of Theorem 4 for this k and then we prove Theorem 3 for $k+1$.

3.2. Let Theorem 3 be proved for a certain k $(\geqq 2)$. Let us consider

$$\overline{\mathscr{F}}=\{X\colon X\subseteq R,\ |X|=d,\ X\notin\mathscr{F}\}\quad\text{and}$$

$$\overline{\mathscr{G}^*_{s,d}(\mathscr{F})}=\{X\colon X\subseteq R,\ |X|=s,\ X\notin\mathscr{G}^*_{s,d}(\mathscr{F})\}.$$

If X is an arbitrary element of $\overline{\mathscr{G}^*_{s,d}(\mathscr{F})}$, then there are no k sets $X_1, X_2, \ldots, X_k\in\mathscr{F}$ such that X is the union of these X_i's. Now we apply Theorem 3. At most $\binom{s-1}{d}$ d-element subsets of X belong to $\mathscr{F}$, i.e., at least $\binom{s}{d}-\binom{s-1}{d}$ d-element subsets of X belong to $\overline{\mathscr{F}}$. On the other hand, each d-element subset of R is contained in exactly $\binom{r-d}{s-d}$ s-element subsets of R.

Hence,

$$\overline{|\mathcal{G}^*_{s,d}(\mathcal{F})|}\left(\binom{s}{d}-\binom{s-1}{d}\right) \leqq |\overline{\mathcal{F}}|\binom{r-d}{s-d},$$

$$\left(\binom{r}{s}-|\mathcal{G}^*_{s,d}(\mathcal{F})|\right)\binom{s-1}{d-1} \leqq \binom{r-d}{s-d}\left(\binom{r}{d}-|\mathcal{F}|\right) \quad \text{and (1).}$$

3.3. Let $d \geqq \frac{r}{k}$. We assume the statement of Theorem 3 is false, i.e., there is a family $\mathcal{F}\in\mathfrak{G}_{d,d}(r,k)$ with $|\mathcal{F}|>\binom{r-1}{d}$. Since $(k-1)d\geqq r-d$ $\mathcal{G}^*_{r-d,d}(\mathcal{F})$ exists and we may use Corollary 2 and obtain

$$|\mathcal{G}^*_{r-d,d}(\mathcal{F})| \geqq \binom{r-1}{r-d}.$$

Consider $\mathcal{G}^{**}_{r-d,d}(\mathcal{F})=\{X\colon R\setminus X\in\mathcal{G}^*_{r-d,d}(\mathcal{F})\}$. Since $\mathcal{G}^{**}_{r-d,d}(\mathcal{F})\cap\mathcal{F}=\varnothing$ (otherwise there would exist k sets of $\mathcal{F}$ with union R) we get the contradiction

$$\binom{r}{d} \geqq |\mathcal{F}|+|\mathcal{G}^{**}_{r-d,d}(\mathcal{F})| > \binom{r-1}{d}+\binom{r-1}{d-1}=\binom{r}{d} \quad \blacksquare$$

4. Proof of theorem 1

Let $\frac{r-1}{2}>d\geqq\frac{r-1}{k-1}$. Let $\mathcal{F}$ be a maximal family of $\mathfrak{G}_{c,d}(r,k)$. Finally let p_i be the parameters of $\mathcal{F}$.

4.1. First we prove the statement of Theorem 1 for $d=\left]\frac{r-1}{k-1}\right[$ ($]x[$ denotes the least integer not less than x). If $p_d>\frac{r-d-1}{r-1}\binom{r}{d}$, then the set $\mathcal{G}^*_{r-1,d}(\mathcal{F}_d)$ of subsets of R which are the union of at most $k-1$ sets of $\mathcal{F}_d$ satisfies

$$|\mathcal{G}^*_{r-1,d}(\mathcal{F}_d)| \geqq 1,$$

by Corollary 3. Without loss of generality let $R\setminus\{r\}\in\mathcal{G}^*_{r-1,d}(\mathcal{F}_d)$. Then there is no set belonging to $\mathcal{F}$ which contains the element r. Otherwise there exist at most k sets having the union R. Thus, $\mathcal{F}$ is a Sperner family on $R\setminus\{r\}$ and the LYM inequality yields the desired result immediately.

If $p_d\leqq\frac{r-d-1}{r-1}\binom{r}{d}$, let us consider $\mathcal{S}\mathcal{F}=\mathcal{F}_0\cup\mathcal{F}_1$, where

$$\mathcal{F}_0 = \{X\colon X\in\mathcal{S}\mathcal{F},\ r\notin X\},$$

and

$$\mathcal{F}_1 = \{X\colon X\in\mathcal{S}\mathcal{F},\ r\in X\}.$$

$\mathscr{F}_0$ is a Sperner family on $R\setminus\{r\}$ and by means of the LYM inequality we have

$$\frac{p_d}{\binom{r-1}{d}}+\frac{|\mathscr{F}_0|-p_d}{\binom{r-1}{d-1}}\leqq 1,$$

and

$$|\mathscr{F}_0|\leqq\binom{r-1}{d-1}+p_d\frac{r-2d}{r-d}\leqq\binom{r-1}{d}-\frac{r-2d}{(r-d)^2}\binom{r-2}{d-1}.$$

$\mathscr{F}_1'=\{X\colon X\cup\{r\}\in\mathscr{F}_1,\ r\notin X\}$ is a Sperner family on $R\setminus\{r\}$ with $|X|\leqq e-1=\left[\frac{r-1}{k}\right]-1$ for all $X\in\mathscr{F}_1'$ which follows by Greene and Hilton's result. Thus,

$$|\mathscr{F}_1|=|\mathscr{F}_1'|\leqq\binom{r-1}{e-1}.$$

The statement of the theorem follows if

$$\frac{r-2d}{(r-d)^2}\binom{r-2}{d-1}\geqq\binom{r-1}{e-1},$$

which is equivalent to

$$Q(r,k)=\frac{r-2d}{(r-d)(r-1)}\prod_{i=0}^{d-e-1}\frac{r-d+1+i}{e+i}\geqq 1.$$

Since

$$Q(r,k)\geqq\frac{r-2d}{(r-d)(r-1)}\left(\frac{r-e}{d-1}\right)^{d-e}\geqq\frac{1}{3(r-1)}\left(\frac{(k-1)^2}{k}\right)^{\frac{r-1}{k(k-1)}}$$

we have $Q(r,k)\geqq 1$ for sufficiently large r (e.g. for $r\geqq 7k(k-1)+1$).

2. Now let $d>d_0=\left]\frac{r-1}{k-1}\right[$. We prove the statement of Theorem 1 by induction on d. If $\mathscr{F}\in\mathfrak{G}_{c,d}(r,k)$, then $\mathscr{F}'=(\mathscr{F}\setminus\mathscr{F}_d)\cup\Delta\mathscr{F}_d$ is a Sperner family and moreover it can be easily verified that $\mathscr{F}'\in\mathfrak{G}_{c,d-1}(r,k)$. By induction hypothesis

$$|\mathscr{F}'|\leqq\binom{r-1}{d-1}.$$

Theorem 3 yields $|\mathscr{F}_d|\leqq\binom{r-1}{d}$ and the results of section 2 imply

$$|\mathscr{F}|=|\mathscr{F}'|+|\mathscr{F}_d|-|\Delta\mathscr{F}_d|\leqq\binom{r-1}{d-1}+|\mathscr{F}_d|\left\{1-\frac{\binom{r-1}{d-1}}{\binom{r-1}{d}}\right\}$$

and, using $\binom{r-1}{d-1} < \binom{r-1}{d}$,

$$|\mathcal{F}| \leqq \binom{r-1}{d-1} + \binom{r-1}{d}\left\{1 - \frac{\binom{r-1}{d-1}}{\binom{r-1}{d}}\right\} = \binom{r-1}{d}.$$

Finally, we notice that $\{X\colon X \subseteqq R,\ |X| = d,\ r \notin X\}$ belongs to $\mathfrak{G}_{c,d}(r, k)$ in all cases. ∎

5. Proof of theorem 2

Let $\frac{r-1}{k-1} > d > \frac{r-1}{k} \geqq c$.

Since $\mathcal{F}' = \{X\colon X \subseteqq R,\ |X| = d,\ r \notin X\} \in \mathfrak{G}_{c,d}(r, k)$ if $c \geqq r-(k-1)d$ and $\mathcal{F}' \cup \{X\colon X \subseteqq R,\ |X| = r-(k-1)d-1,\ r \in X\} \in \mathfrak{G}_{c,d}(r, k)$ if $c \leqq r-(k-1)d-1$ it is sufficient to prove for arbitrary $\mathcal{F} \in \mathfrak{G}_{c,d}(r, k)$:

$$|\mathcal{F}| \leqq \begin{cases} \binom{r-1}{d} + \binom{r-1}{r-(k-1)d-2} & \text{if } c \leqq r-(k-1)d-1, \\ \binom{r-1}{d} & \text{if } c \geqq r-(k-1)d, \end{cases}$$

for $r \geqq r_0(k, \varepsilon)$ and $d = \frac{r-1}{k-1+\varepsilon}$ with fixed ε, $0 < \varepsilon < 1$.

Let $\mathcal{F}$ be an arbitrary family of $\mathfrak{G}_{c,d}(r, k)$. Consider $\mathcal{S}\mathcal{F}$. $\mathcal{S}\mathcal{F}$ is a Sperner family but not necessarily a family of $\mathfrak{G}_{c,d}(r, k)$. We decompose $\mathcal{F}$ in the subfamilies $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{H}$, defined as follows:

$\mathcal{D}$ is a subfamily with $\mathcal{S}\mathcal{D} = \{X\colon X \in \mathcal{S}\mathcal{F},\ r \notin X\}$,

$\mathcal{E} = \{X\colon X \in \mathcal{F} \setminus \mathcal{D},\ |X| \leqq r-(k-1)d-1\}$,

$\mathcal{H} = \{X\colon X \in \mathcal{F} \setminus \mathcal{D},\ |X| \geqq r-(k-1)d\}$.

5.1. $\mathcal{S}\mathcal{D}$ is a Sperner family on $R \setminus \{r\}$ and the LYM inequality yields

$$\sum_{X \in \mathcal{S}\mathcal{D}} \frac{1}{\binom{r-1}{|X|}} \leqq 1,$$

$$\frac{p_d}{\binom{r-1}{d}} + \frac{|\mathcal{S}\mathcal{D}| - p_d}{\binom{r-1}{d-1}} \leqq 1$$

and

$$|\mathcal{D}| = |\mathcal{S}\mathcal{D}| \leqq \frac{d}{r-d}\binom{r-1}{d} + \frac{r-2d}{r-d}\, p_d.$$

5.2. If $c \geqq r-(k-1)d$, then $|\mathscr{E}|=0$. If $c \leqq r-(k-1)d-1$, then

$$\mathscr{J}=\{X\colon X\cup\{r\}\in\mathscr{S}(\mathscr{D}\cup\mathscr{E}),\ r\notin X\}$$

is a Sperner family on $R\setminus\{r\}$ with cardinality $|\mathscr{E}|$ and with $|X|\leqq r-(k-1)d-2$ for all $X\in\mathscr{J}$. The LYM inequality yields

$$\sum_{X\in\mathscr{J}}\frac{1}{\binom{r-1}{|X|}}\leqq 1,$$

and

$$|\mathscr{E}|=|\mathscr{J}|\leqq\binom{r-1}{r-(k-1)d-2}.$$

5.3. If $\min_{X\in\mathscr{D}}|X|\leqq r-(k-1)d-1$, then $|\mathscr{H}|=0$ and the statement follows immediately. Let $|X|\geqq r-(k-1)d$ for all $X\in\mathscr{D}$.

Let $\mathscr{G}^{*}_{(k-1)d,d}(\mathscr{F}_d)$ be the set of all $(k-1)d$-element subsets of R which are the union of at most $(k-1)$ sets of $\mathscr{F}_d$. Furthermore, let

$$\mathscr{G}^{**}_{(k-1)d,d}(\mathscr{F}_d)=\{X\colon R\setminus X\in\mathscr{G}^{*}_{(k-1)d,d}(\mathscr{F}_d)\}.$$

Then $\mathscr{D}\cup\mathscr{H}\cup\mathscr{G}^{**}_{(k-1)d,d}(\mathscr{F}_d)$ is a Sperner family too. Clearly, $\mathscr{D}\cup\mathscr{H}$ and $\mathscr{G}^{**}_{(k-1)d,d}(\mathscr{F}_d)$ are Sperner families themselves. We notice that

$$|X|\geqq r-(k-1)d \quad\text{for all}\quad X\in\mathscr{D}\cup\mathscr{H} \quad\text{and}$$
$$|X|=r-(k-1)d \quad\text{for all}\quad X\in\mathscr{G}^{**}_{(k-1)d,d}(\mathscr{F}_d).$$

We have only to show that there is no pair (Y,Z) with $Y\in\mathscr{G}^{**}_{(k-1)d,d}(\mathscr{F}_d)$ and $Z\in\mathscr{D}\cup\mathscr{H}$ satisfying $Y\subseteqq Z$. Assume the contrary. Then there are $k-1$ sets $X_1,\dots,X_{k-1}$ belonging to $\mathscr{F}_d$ satisfying $R\setminus Y=\bigcup_{i=1}^{k-1}X_i$. Hence, $X_1,\dots,X_{k-1}$ and Z, all sets belonging to $\mathscr{F}$, have the union R, in contradiction to our assumption.

$$\mathscr{J}'=\{X\colon X\cup\{r\}\in\mathscr{S}(\mathscr{D}\cup\mathscr{H}\cup\mathscr{G}^{**}_{(k-1)d,d}(\mathscr{F}_d)),\ r\notin X\}$$

is a Sperner family on $R\setminus\{r\}$. If q_i, q_i' and q_i'' are the parameters of the families $\mathscr{J}'$, $\mathscr{H}$ and $\mathscr{G}^{**}_{(k-1)d,d}(\mathscr{F}_d)$, respectively, then $q_i=q'_{i+1}+q''_{i+1}$. By Greene and Hilton's result (see section 2) we know that the cardinality of the sets of $\mathscr{J}'$ are upperbounded by $e=\left[\frac{r-1}{k}\right]-1$. The LYM inequality, Theorem 4 and simple estimations of binomial coefficients yield

$$\sum_{X\in\mathscr{J}'}\frac{1}{\binom{r-1}{|X|}}\leqq 1,$$

$$\frac{|\mathscr{H}|}{\binom{r-1}{e-1}}+\frac{|\mathscr{G}^{**}_{(k-1)d,d}(\mathscr{F}_d)|}{\binom{r-1}{r-(k-1)d-1}}\leqq 1$$

and

$$|\mathcal{H}| \leqq \binom{r-1}{e-1}\left\{1-\frac{(k-1)d}{r-(k-1)d}\left(\frac{r-d}{d}\frac{|\mathcal{F}_d|}{\binom{r-1}{d}}+\frac{r}{(k-1)d}-\frac{r}{d}\right)\right\}.$$

Thus, $|\mathcal{F}|=|\mathcal{D}|+|\mathcal{E}|+|\mathcal{H}|\leqq f(p_d)+|\mathcal{E}|$, where

$$f(p_d) = \frac{d}{r-d}\binom{r-1}{d}+\frac{r-2d}{r-d}p_d+$$

$$+\binom{r-1}{e-1}\left\{1-\frac{(k-1)d}{r-(k-1)d}\left(\frac{r-d}{d}\frac{p_d}{\binom{r-1}{d}}+\frac{r}{(k-1)d}-\frac{r}{d}\right)\right\}.$$

Since 5.2. it is sufficient to prove $f(p_d)\leqq\binom{r-1}{d}$ for all p_d with $0\leqq p_d\leqq\binom{r-1}{d}$ (see Theorem 3). $p_d=\binom{r-1}{d}$ implies $f(p_d)=\binom{r-1}{d}$. $f(p_d)$ is a linear function in p_d. Hence, it is sufficient to show that the factor of p_d is nonnegative, i.e.

$$\frac{r-2d}{r-d} \geqq \frac{(k-1)d}{r-(k-1)d}\frac{r-d}{d}\frac{\binom{r-1}{e-1}}{\binom{r-1}{d}},$$

or equivalently

$$Q(r, k, d) = \frac{(r-2d)(r-(k-1)d)}{(r-d)^2(k-1)}\frac{\binom{r-1}{d}}{\binom{r-1}{e-1}} \geqq 1.$$

Then

$$Q(r, k, d) = \frac{(r-2d)(r-(k-1)d)}{(r-d)^2(k-1)}\prod_{i=0}^{d-e}\frac{r-d+i}{e+i} \geqq$$

$$\geqq \frac{1}{4r}\left(\frac{(k-1)^2}{k}\right)^{\frac{1-\varepsilon}{k^2}(r-1)} \geqq \frac{1}{4r}(k-2)^{(1-\varepsilon)\frac{r-1}{k^2}}$$

and $Q(r, k, d)\geqq 1$ for sufficiently large r $\Bigg($e.g. $r\geqq\left(\frac{5k^2}{1-\varepsilon}\right)^2+1$, since $\frac{r-1}{r}\geqq\frac{4}{5}$ and $2^x>\frac{x^2}{5}$ for nonnegative real numbers we obtain

$$Q(r, k, d) \geqq \frac{1}{4r}2^{(1-\varepsilon)\frac{r-1}{k^2}} > \frac{1}{20r}\left((1-\varepsilon)\frac{r-1}{k^2}\right)^2 \geqq \frac{1}{25}(r-1)\left(\frac{1-\varepsilon}{k^2}\right)^2 \geqq 1\Bigg). \quad \blacksquare$$

6. Concluding remarks

6.1. In [11], where the author discussed the case $k=3$, our method was presented in a more refined version. The proofs are complicated and extensive there. Using this refined method our results in Theorem 1 and 2 could be improved (i.e. the r_0's) surely, but the proof would be much more laborious than that of [11].

A very important structural assertion on Sperner families is that every Sperner family has a *canonical* Sperner family with the same parameters (see section 2). If $\mathcal{F}\in\mathfrak{G}_{c,d}(r,k)$, then $\mathcal{S}\mathcal{F}$ does not belong to $\mathfrak{G}_{c,d}(r,k)$ in general. See e.g. the case $k=3$. Let $r\geqq 7$. Let $\mathcal{F}$ consist of the sets $\{1,2,\ldots,r-3\}$, $\{i,r-2\}$, $\{i,r-1\}$ and $\{i,r\}$ with $i=1,2,\ldots,r-3$. Then $\mathcal{F}\in\mathfrak{G}_{0,r}(r,3)$. $\mathcal{S}\mathcal{F}$ consists of the sets

$$\{1,2,\ldots,r-3\},$$

$$\{i,r-2\}\quad (i=1,2,\ldots,r-3),$$

$$\{i,r-1\}\quad (i=1,2,\ldots,r-2)\quad\text{and}$$

$$\{i,r\}\quad (i=1,2,\ldots,r-4).$$

The 3 sets $\{1,2,\ldots,r-3\}$, $\{r-2,r-1\}$, and $\{1,r\}$, all sets of $\mathcal{S}\mathcal{F}$, have the union R.

It remains an open

Problem. *Under what conditions can be stated that* $\mathcal{F}\in\mathcal{G}_{c,d}(r,k)$ *implies* $\mathcal{S}\mathcal{F}\in\mathcal{G}_{c,d}(r,k)$?

6.2. Using Corollary 3 we are able to find, for small r, a new upper bound for the maximum cardinality of the k-uniform, not l-intersecting families. More precisely, let $n(r,k,l)$ denote the maximum size of a family of k-element subsets of R, $|R|=r$, such that $|X\cap Y|\neq l$ for all $X,Y\in\mathcal{F}$. If $k\leqq l$ or $2k-l>r$, then obviously $n(r,k,l)=\binom{r}{k}$. Let $k<l$ and $r\geqq 2k-l$. As special cases, Ray-Chaudhuri and Wilson [18] proved

$$n(r,k,l)\leqq\binom{r}{k-1},\tag{2}$$

and Deza, Erdős, Frankl [3] resp. Frankl [6], [7] determined, for $r>r_0(k)$ better and best upper bounds for $n(r,k,l)$. $n(r,k,0)$ is determined in the Erdős—Ko—Rado theorem [4], while the extremal families are described in Hilton and Milner [13].

Theorem 5. (i) *If* $r\geqq 2k\geqq 2$ *then* $n(r,k,0)=\binom{r-1}{k-1}$.

(ii) $\mathcal{F}$ *is a maximal if and only if*

a) $r\neq 2k$: $\mathcal{F}$ *consists of all k-element subsets of R containing a fixed element* $v\in R$.

b) $r=2k$: $\mathcal{F}$ *contains exactly one set of every pair of complementary sets.*

We are able to improve (2) for small r, namely for

$$2k-l \leqq r < 3k+l-1-\frac{l^2}{k-l}.$$

Theorem 6. (i) *Let* $k>l\geqq 1$ *and* $r\geqq 2k-l$, *then*

$$n(r,k,l) \leqq \frac{k-l}{2k-l}\binom{r}{k}. \tag{3}$$

(ii) *If* $r=2k-l$ *then* (3) *is sharp.* $\mathcal{F}$ *is maximal if and only if* $\mathcal{F}$ *consists of all k-element subsets of* $R-\{v\}$, *for some fixed* $v\in R$.

Proof. Let $\mathcal{F}$ be a maximal k-uniform, not l-intersecting family, $|\mathcal{F}|=n(r,k,l)$, $k>l\geqq 1$, $r\geqq 2k-l$.

(i) The condition $|X\cap Y|\neq l$ means $\mathcal{G}^*_{2k-l,k}(\mathcal{F})=\emptyset$. Corollary 3 implies (3).

(ii) Let $r=2k-l$. Consider $\bar{\mathcal{F}}=\{X\colon R\setminus X\in\mathcal{F}\}$. Obviously, $\bar{\mathcal{F}}$ is an $(r-k)$-uniform, not 0-intersecting family. Also Theorem 5 yields

$$|\mathcal{F}| = |\bar{\mathcal{F}}| \leqq \binom{r-1}{r-k-1} = \binom{r-1}{k}.$$

Since $r=2k-l\neq 2k$, the maximal $\bar{\mathcal{F}}$'s are described in Theorem 5 (ii)a. The corresponding $\mathcal{F}$'s are the desired ones. Finally we note that $\frac{k-l}{2k-l}\binom{r}{k}<\binom{r}{k-1}$ if and only if $r<3k+l-1-\frac{l^2}{k-l}$. ∎

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